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MATHEMATICS

Convex Polyhedra with Parquet Faces

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In addition to the solids with regular faces found nearly 50 years ago [1], another 76 polyhedra with conditional edges joining regular polygons constituting a single face are on the recently compiled list of all convex polyhedra with a positive curvature at each vertex and with each face composed of convex regular polygons or being such a polygon [2, 3]. Below, each polyhedron on this list is referred to as a convex regular-hedron.[1] Applications of regular-hedra and the influence of applications on the development of the theory of polyhedra are addressed in [2]. It should only be added to what was said in [2] that new regular-hedra are of interest to architects and designers, including space architects [10].

In [2, 3] both proofs of the theorem describing the convex regular-hedra are based on the 1973 list of all noncomposite (simple) solids [5–7], i.e., convex regular-hedra that cannot be dissected into two other regular-hedra by a plane. All the composite solids can be obtained by joining the noncomposite polyhedra

$$\Pi_3, \Pi_4, \Pi_5, \Pi_6, \Pi_8, \Pi_{10};$$
$$A_4, A_5, A_6, A_8, A_{10}; \quad M_1, \ldots, M_{15}, M_{20}, M_{22}; Q_4,$$

where the first six are prisms with the subscript indicating the number of sides in the base, the following five are antiprisms, the following 17 are Zalgaller solids, and the last is the Ivanov polyhedron. The solid Q_4 (Fig. 1) can be deleted from this list, but the composite polyhedron $M_3 + Q_4$ is then obtained by joining three solids: two pentagonal pyramids M_3 and the bilunabirotunda M_8.

[1] I am taking this opportunity to note that in [4] the last polyhedron in Proposition 3(iv) is to be deleted, while, in the theorem, the number 151 is to be replaced by 149 or 151 remains, but the word "two" is to be deleted from the definition of a regular-hedron.

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The algorithm used in [3] to produce each composite polyhedron involves a linear representation of the symmetry group of its constituent regular-hedra and explicit coordinate triples of fundamental vertices of these regular-hedra specified as elements of an algebraic extension of the field of rational numbers. To compute the symmetry group of a polyhedron, it is convenient to consider its "skeleton" in the form of an algebraic model whose support consists of the vertex set and whose relation set is composed of the edges, i.e., of unordered pairs of vertices. Thus, in contrast to the purely geometric constructions in [2], the classification proof in [3] combines algebraic, geometric, and computer arguments with the visual monitoring of three-dimensional constructions and computer algebra calculations. To a certain extent, due to this poof, the reader gets rid of the inconveniences associated with the risk of logical gaps or misprints.

The pattern complicates considerably if we assume that there is a vertex with a zero curvature. Such a vertex is called conditional. It can lie on an edge (Fig. 2) or inside a face (as in a prism whose base is composed of two squares and three triangles). Moreover, by Descartes' theorem, the total curvature of all the vertices of a convex polyhedron is 4π and, even if we consider polyhedra with a bounded number of face sides, the

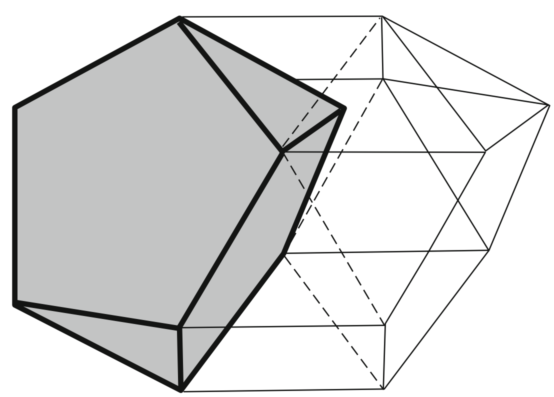

Fig. 1. The tridiminished icosahedron M_7 inside the Ivanov polyhedron Q_4.

number of regular faces can increase in an infinite series of polyhedra with conditional vertices. An example is a series of k prisms $k\Pi_l$ ($k = 1, 2, \ldots$) joined by their bases with a fixed number l of base sides. The lateral faces of $k\Pi_l$ are rectangles composed of k squares. Starting with the second, each polyhedron in this and other similar series can be divided by a plane into polyhedra with regular faces or faces composed of regular polygons. For each polyhedron in the following series, there is no such plane.

Example. The tridiminished icosahedron M_7 (Fig. 1) is cut along edges into two 4-hedral surfaces: two adjacent pentagons with triangles in between and, sharing a vertex, three triangles with a pentagon. Then we separate these surfaces by an edge-length distance along a line perpendicular to the common edge of two pentagons in one surface and to the edge joining the triangle in this surface to a pentagon in the other surface. Now four rhombuses, each composed of two regular triangles, and two squares can be fitted between these surfaces. Adding these six quadrilaterals to two 4-hedral surfaces yields the regular-hedron $M_{7,1}$, which is the Pryakhin polyhedron Q_4 (Fig. 1). Then, again, one of the above 4-hedral surfaces is cut off and shifted by an edge-length distance along the same line. Again, there are four rhombuses (composed of two regular triangles) and two squares between the surfaces. Adding these six quadrilaterals to the tetrahedral and decahedral surfaces making up Q_4 before cutting, we obtain a polyhedron with conditional vertices, which is denoted by $M_{7,2}$. The faces of $M_{7,2}$ that were faces of M_7 before cutting are shown in white in Fig. 2. Continuing to move the 4-hedral surface and add the above six faces at each step, we obtain the polyhedra $M_{7,3}$, $M_{7,4}$, … .

It may seem that the problem of describing all these series is intractable. However, if we put aside the edge lengths but leave the angles the same as in regular polygons, as was shown in [8], convex polyhedra with equiangular faces and faces divided by conditional edges into equiangular ones can be investigated in a similar manner to convex regular-hedra. The fact is that, in addition to four infinite series, there are only a finite number of types of such polyhedra, i.e., polyhedra with a given combinatorial structure and a fixed list of plane angles in the faces at each vertex. Although this was known as early as 35 years ago, the process of describing convex polyhedra with regular faces and faces composed of regular polygons under the assumption of zero-curvature vertices did not begin until last year, when the above-mentioned description of convex regular-hedra was produced. Apparently, the reason is that describing all such polyhedra requires repeating, in a more general situation, the path followed by researchers while creating the list of all noncomposite polyhedra, which took about ten years in the last century [5–7]. At present, the difficulties in this path associated with the large amount of computations and repeated logically indistinguishable arguments can be

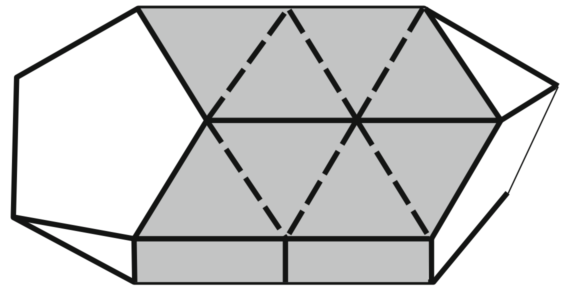

Fig. 2. The 1-regular-hedron $M_{7,2}$.

overcome by using computer algebra and computer graphics systems. Moreover, the series of same-type polyhedra, which seem immense at first glance (see the typical remark in [2, p. 245]), admit an explicit description when constraints are introduced on the number of conditional vertices in each edge of a polyhedron.

Recall that a convex polygon is said to be a parquet one if it can be composed of a finite number of equiangular polygons [8]. The alternation of angles in traversing a parquet k-gon is called its type, which is characterized by the set of numbers (n_1, n_2, ..., n_k). Each number is assigned to a vertex and means that the polygon angle with the vertex n_i equals the angle in a regular n_i-gon. If there are sequential vertices of identical angles, power notation is used. For example, a parquet pentagon composed of a square and a triangle has the type ($3, 12, 4^2, 12$). All 23 types of parquet k-gons, each admitting a partition into more than one equiangular polygon, were indicated in [8]. Moreover, it was found that $k = 3, 4, \ldots, 12$.

Proposition 1. *A parquet polygon of each type can be made up of regular polygons with unit edges. Each edge of such a parquet polygon has a length of one or two, and only the following types of parquet polygons must have edges with different lengths:*

$$(3^2, 6^2);\ (3, 6^4);\ (6^2, 12, 4^2, 12);$$

$$(6^2, 30, 5^3, 30);\ (3, 12^2, 6^2, 12^2);\ (3, 30^2, 6^2, 30, 5, 30);$$

$$(4, 12^2, 4, 12^4);\ (6^2, 30, 5, 30, 6^2, 30^2);$$

$$(6, 12^2, 6, 12^6);\ (6, 12^{10}).$$

Note that some noncomposite polyhedra become composite under the assumption of zero-curvature vertices and parquet faces. Indeed, after conditional vertices appear at the middle of its edges, the noncomposite tetrahedron M_1 is divided by a square section into two equal polyhedra, each consisting of a square pyramid M_2 and two tetrahedra M_1. In turn, M_2 can also be divided by a plane through the midlines of three triangular lateral faces and the base into parts with triangular and parquet faces composed of triangles and squares. This division process can be continued infinitely to obtain increasingly finer pyramids M_1

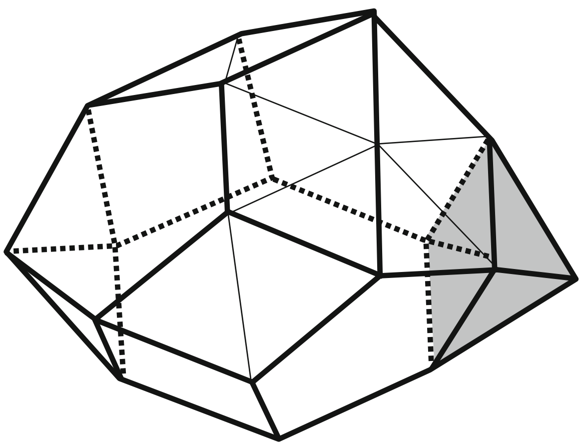

Fig. 3. The polyhedron Q_{6a} joined to the pyramid M_2, whose lateral faces are shaded.

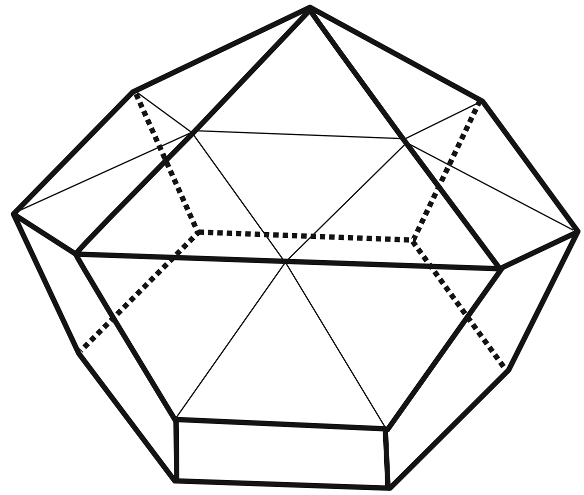

Fig. 4. The polyhedron Q_{6b}.

and M_2 at each step. To describe these and above-discussed series of polyhedra, we introduce the following definition.

Definition. Let m be a nonnegative integer. A convex polyhedron is called an m-regular-hedron if the following conditions are satisfied: (1) the faces are composed of regular polygons with positive integer side lengths; (2) the longest edge has a length of $m + 1$; and (3) any similar polyhedron with a similarity coefficient less than unity has an edge whose length is not a positive integer or has a face that cannot be made up of regular polygons with integer edges.

Thus, each convex regular-hedron is a 0-regular-hedron. Examples of 0-regular-hedra that are not regular-hedra are prisms whose bases consist of parquet polygons of types $(4, 12^2, 4, 12, 6, 12)$, $(6^2, 12^2, 6^2, 12^2)$, $(6, 12^2, 6, 12^2, 6, 12^2)$, and $(6, 12^4, 6, 12^4)$ and whose lateral faces are squares. The series of polyhedra M_7, $M_{7,1} = Q_4$, $M_{7,2}$, ... consists of two regular-hedra and m-regular-hedra, $m = 1, 2, \ldots$.

Definition. Let m_0 be a nonnegative integer. A convex m_0-regular-hedron is said to be composite if some plane divides it into an m_1-regular-hedron and an m_2-regular-hedron, where m_1 and m_2 are nonnegative integers. Otherwise, the convex m_0-regular-hedron is called noncomposite.

Theorem. *Except for seven polyhedra, each noncomposite regular-hedron with unit edges is a noncomposite* 0*-regular-hedron. The exceptions can be represented as the following joins of m-regular-hedra, m* = 0, 1.

(i) *The prism* Π_6 *consists of two prisms with* $(3^2, 6^2)$-*type trapezoidal bases joined along the rectangular faces,* $\Pi_6 = 3\Pi_3$.

(ii) *The triangular cupola* $M_4 = (P_{2,25} + M_1) + (P_{2,25} + P_{2,25}) = 4M_1 + 3M_2$, *where* $P_{2,25} = M_1 + M_2$ *is the join of the tetrahedron* M_1 *and the square pyramid* M_2 *by lateral faces* [3].

(iii) *The truncated tetrahedron* $M_{10} = (3M_1 + 2M_2) + (M_4 + 3M_2) = 7M_1 + 8M_2$.

(iv) *The truncated octahedron* $M_{16} = 2M_{2a} + M_{16a}$, *where* M_{2a} *is a quadrilateral pyramid with a square base of side length* 2 *truncated at the midlines of the lateral faces and* M_{16a} *is the* 1*-regular-hedron remaining after two solids* M_{16} *have been cut off from* M_{2a} *by parallel planes.*

(v) *The truncated icosahedron* $M_{19} + 2M_{3a} + M_{19a} = 3M_{3a} + M_{19b}$, *where* M_{3a} *is a pentagonal pyramid with a regular pentagonal base of side length* 2 *truncated at the midlines of the lateral faces,* M_{19a} *is the* 1*-regular-hedron remaining after two solids* M_{19} *have been cut off from* M_{19} *by parallel planes of two solids* M_{3a} *and* M_{19b} *is the* 1*-regular-hedron remaining after three solids* M_{3a} *have been cut off from* M_{19} *by three planes.*

(vi) *The oblique prism* (*Ivanov polyhedron*) $Q_1 = 6M_1 + 6M_2$.

(vii) *The Ivanov polyhedron* $Q_2 = 16M_1 + 16M_2$.

Proposition 2. *The noncomposite* 1*-regular-hedra consist of the solids* M_{2a}, M_{3a}, M_{16a}, M_{19a}, *and* M_{19b} *mentioned in the theorem; the solids* Q_{6a} (Fig. 3) *and* Q_{6b} (Fig. 4)*, which are obtained from the Ivanov polyhedron* Q_6 *in the same way as the latter is obtained from the triangular hebesphenorotunda* M_{20} [7, 9]; *and the antiprism B obtained from the antiprism* A_k *with edge length* 2 *and a k-angular base (k* = 4, 5, ...*) by cutting the latter with a plane through the midlines of the side triangles* [8].

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